Nonstandard Models of Fragments of Church's Type Theory

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Simple Types

Simple Types $\mathcal{T}$:

o	(truth values)
ι	(individuals)
$(\alpha\beta)$	(functions from β to α)

$(\alpha\beta\gamma)$ abbreviates $((\alpha\beta)\gamma)$

A Standard Frame:

$$\begin{array}{lcl} \mathcal{D}_o & = & \{\mathtt{T},\mathtt{F}\}. \\ \mathcal{D}_\iota & = & \mathbb{N} \text{ (natural numbers).} \\ \mathcal{D}_{\alpha\beta} & = & \mathcal{D}_\alpha^{\mathcal{D}_\beta}\text{, all functions from } \mathcal{D}_\beta \text{ to } \mathcal{D}_\alpha. \end{array}$$

$\mathcal{D}_{o\iota} \cong \mathcal{P}(\mathbb{N})$: $X \subseteq \mathbb{N} \leftrightarrow \chi_X : \mathbb{N} \to \{\mathtt{T},\mathtt{F}\}$ (characteristic functions)

$\mathcal{D}_{o\iota\iota} \cong \mathcal{P}(\mathbb{N} \times \mathbb{N})$: Binary relations on $\mathbb{N}$

$\mathcal{D}_{o(o\iota)} \cong \mathcal{P}(\mathcal{P}(\mathbb{N}))$

Simply Typed λ-Calculus

	x_α	Variables ($\mathcal{V}$)
	A_α	Parameters ($\mathcal{P}$)
Terms:	c_α	Logical Constants ($\mathcal{S}$)
	$[\mathbf{F}_{\alpha\beta}\,\mathbf{B}_\beta]_\alpha$	Application
	$[\lambda y_\beta\,\mathbf{A}_\alpha]_{\alpha\beta}$	λ-abstraction

Conversion and Reduction

Equality of terms: $\alpha\beta\eta$

α-conversion	Changing Bound Variables	
β-reduction	$[[\lambda y_\beta \, \mathbf{A}_\alpha] \, \mathbf{B}] \xrightarrow{\beta} [\mathbf{B}/y]\mathbf{A}$	
η-reduction	$[\lambda y_\beta \, [\mathbf{F}_{\alpha\beta} \, y]] \xrightarrow{\eta} \mathbf{F}$	$(y_\beta \notin \mathbf{Free}(\mathbf{F}))$

Every term has a unique $\beta\eta$-normal form,

up to α-conversion.

Logical Constants

Some logical constants which may be in $\mathcal{S}$:

$\neg_{oo}$	negation
$\vee_{ooo}$	disjunction
$\wedge_{ooo}$	conjunction
$=^{\alpha}_{o\alpha\alpha}$	equality at type α
$\Pi^{\alpha}_{o(o\alpha)}$	universal quantification over type α
$\Sigma^{\alpha}_{o(o\alpha)}$	existential quantification over type α

Logical Constants

Notation:

$$[\mathbf{A}_o \lor \mathbf{B}_o] \quad \text{means} \quad [\lor_{ooo} \mathbf{A}_o \mathbf{B}_o]$$

$$[\mathbf{A}_o \land \mathbf{B}_o] \quad \text{means} \quad [\land_{ooo} \mathbf{A}_o \mathbf{B}_o]$$

$$[\mathbf{A}_\alpha =^\alpha \mathbf{B}_\alpha] \quad \text{means} \quad [=^\alpha_{o\alpha\alpha} \mathbf{A}_\alpha \mathbf{B}_\alpha]$$

$$[\forall x_\alpha \mathbf{A}_o] \quad \text{means} \quad [\Pi^\alpha_{o(o\alpha)} [\lambda x_\alpha \mathbf{A}_o]].$$

$$[\exists x_\alpha \mathbf{A}_o] \quad \text{means} \quad [\Sigma^\alpha_{o(o\alpha)} [\lambda x_\alpha \mathbf{A}_o]].$$

Church's Type Theory

Church's Type Theory:

- Simply typed λ-calculus with the signature $\{\neg, \vee\} \cup \{\Pi^{\alpha} \mid \alpha \in \mathcal{T}\}$ (and perhaps a description or choice operator).
- Axioms of Extensionality
- Axiom of Description or Choice
- Axiom of Infinity

Extensional Type Theory

$\mathcal{S}$ Fragment of Extensional Type Theory:

- Simply typed λ-calculus with the signature $\mathcal{S}$
- Extensionality
- No Axiom of Description or Choice
- No Axiom of Infinity

Surjective Cantor Theorem

There is no surjection from $\mathcal{D}_\iota$ onto $\mathcal{D}_{o\iota}$.

$$\neg\exists\, g_{o\iota\iota}\,\forall\, f_{o\iota}\,\exists\, x_\iota\,[g\, x \;=^{o\iota}\; f]$$

Suppose $G_{o\iota\iota}$ is a surjection.

- Let $D_{o\iota}$ be $[\lambda x_\iota\,\neg\,[G\,x\,x]]$ (diagonal set).

$$\{x \,\Big|\, x \notin [G\,x]\}$$

- There is some X_ι such that $G\,X \;=^{o\iota}\; D$.

$$G\,X\,X \;\equiv\; D\,X \;\equiv\; \neg\,[G\,X\,X]$$

Contradiction

Instantiating Set Variables in Automated Theorem Proving

When a higher-order automated theorem prover (e.g., Tps) searches for a proof of a theorem, it must synthesize instantiations for set variables using logical constants.

The diagonal set

$$[\lambda x_\iota \neg [G\, x\, x]]$$

(using $\neg$) is a simple example.

Synthesizing sets using logical constants is hard in general.

Instantiating Set Variables in Automated Theorem Proving

Questions:

- Do we really need to consider logical constants?
- Can we prove the surjective Cantor theorem without using logical constants like negation?

Answers:

- Yes
- No

Surjective Cantor Theorem

Goal: Prove the surjective Cantor theorem is not a theorem of the $\mathcal{S}$ fragment of extensional type theory for some $\mathcal{S}$.

Note: If $\neg \in \mathcal{S}$, then we can define the diagonal set. So, $\neg$ will not be in $\mathcal{S}$.

Method: Construct a combinatory frame $\mathcal{D}$ such that

$$\exists\, g \in \mathcal{D}_{o\iota\iota}\, \forall\, f \in \mathcal{D}_{o\iota}\, \exists\, x \in \mathcal{D}_{\iota}\, (g\, x \,=\, f)$$

Frames in General

$$\begin{array}{lcl} \mathcal{D}_o & = & \{\mathtt{T}, \mathtt{F}\} \\ \mathcal{D}_\iota & = & \text{any nonempty set} \\ \mathcal{D}_{\alpha\beta} & \subseteq & (\mathcal{D}_\alpha)^{\mathcal{D}_\beta} \text{ (maybe not all functions)} \end{array}$$

To interpret λ-abstractions, $\mathcal{D}_{\alpha\beta}$ must contain "enough" functions from $\mathcal{D}_\beta$ to $\mathcal{D}_\alpha$. Such frames are combinatory.

Frames Realizing Logical Constants

$$\begin{array}{lcl} \mathcal{D}_o & = & \{\mathtt{T}, \mathtt{F}\} \\ \mathcal{D}_\iota & = & \text{any nonempty set} \\ \mathcal{D}_{\alpha\beta} & \subseteq & (\mathcal{D}_\alpha)^{\mathcal{D}_\beta} \text{ (maybe not all functions)} \end{array}$$

We may want $\mathcal{D}$ to include interpretations for logical constants.

• We say $\mathcal{D}$ realizes a logical constant c_α if there is an *appropriate* interpretation for c in $\mathcal{D}_\alpha$.

Example:

• $\mathcal{D}$ realizes $\neg$ if the negation function $\neg : \mathcal{D}_o \longrightarrow \mathcal{D}_o$ is in $\mathcal{D}_{oo}$.

$\mathcal{S}$-Models

$$\begin{array}{lcl} \mathcal{D}_o & = & \{\mathtt{T}, \mathtt{F}\} \\ \mathcal{D}_\iota & = & \text{any nonempty set} \\ \mathcal{D}_{\alpha\beta} & \subseteq & (\mathcal{D}_\alpha)^{\mathcal{D}_\beta} \text{ (maybe not all functions)} \end{array}$$

Let $\mathcal{S}$ be a signature of logical constants.

- We say $\mathcal{D}$ realizes $\mathcal{S}$ if $\mathcal{D}$ realizes every $c \in \mathcal{S}$.
- An $\mathcal{S}$-model (of the $\mathcal{S}$ fragment of extensional type theory) is a combinatory frame $\mathcal{D}$ realizing $\mathcal{S}$.

There is a proof system $\vdash_{\mathcal{S}} \mathbf{M}$ for closed propositions $\mathbf{M}$.

$\mathcal{S}$-models are sound and complete with respect to $\vdash_{\mathcal{S}}$.

(Binary) Logical Relation Frames

Start with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$, $\mathcal{D}_\iota$ nonempty,

$\mathcal{R}_o \subseteq (\mathcal{D}_o)^2$ (e.g., $\leq^o$) and $\mathcal{R}_\iota \subseteq (\mathcal{D}_\iota)^2$.

Assume $\mathcal{R}_o$ and $\mathcal{R}_\iota$ are reflexive binary relations.

Extend to function types:

Assume $\mathcal{D}_\alpha$, $\mathcal{R}_\alpha \subseteq (\mathcal{D}_\alpha)^2$, $\mathcal{D}_\beta$ and $\mathcal{R}_\beta \subseteq (\mathcal{D}_\beta)^2$ are defined.

Define $\mathcal{D}_{\alpha\beta}$:

$$\{f : \mathcal{D}_\beta \to \mathcal{D}_\alpha \,\big|\, \forall \langle y^0, y^1 \rangle \in \mathcal{R}_\beta \Rightarrow \langle f(y^0), f(y^1) \rangle \in \mathcal{R}_\alpha\}$$

Define $\mathcal{R}_{\alpha\beta} \subseteq (\mathcal{D}_{\alpha\beta})^2$:

$$\{\langle f^0, f^1 \rangle \,\big|\, \forall \langle y^0, y^1 \rangle \in \mathcal{R}_\beta \Rightarrow \langle f^0(y^0), f^1(y^1) \rangle \in \mathcal{R}_\alpha\}$$

Logical Relation Frames

Fact 1: Each $\mathcal{R}_\alpha \subseteq \mathcal{D}_\alpha^2$ is reflexive.

Fact 2: $\mathcal{D}$ is a combinatory frame.

Construction works with relations of any arity.

Let A be a nonempty set.

function $p : A \to \mathcal{D}_\alpha$	$\sim$	A-tuple $\langle p(i) \rangle_{i \in A}$ of elements of $\mathcal{D}_\alpha$
set of functions $\mathcal{R}_\alpha \subseteq \mathcal{D}_\alpha^A$	$\sim$	A-ary relation on $\mathcal{D}_\alpha$

Logical Relation Frames

Let $x \in \mathcal{D}_\alpha$ and $K_x : A \to \mathcal{D}_\alpha$ be the

constant function $K_x(i) = x$ for all $i \in A$.

constant function K_x $\sim$ A-tuple $\langle x \rangle_{i \in A}$.

Instead of *reflexivity* of $\mathcal{R}_\alpha$,

we will need all constant functions K_x to be in $\mathcal{R}_\alpha$.

Generalizing Logical Relation Frames

Start with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$, $\mathcal{D}_\iota$ nonempty, $\mathcal{R}_o \subseteq \mathcal{D}_o^{\boxed{2}}$ and $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^{\boxed{2}}$.

Assume $\mathcal{R}_o$ and $\mathcal{R}_\iota$ include constant functions.

Extend to function types:

Assume $\mathcal{D}_\alpha$, $\mathcal{R}_\alpha \subseteq (\mathcal{D}_\alpha)^{\boxed{2}}$, $\mathcal{D}_\beta$ and $\mathcal{R}_\beta \subseteq (\mathcal{D}_\beta)^{\boxed{2}}$ are defined.

Define $\mathcal{D}_{\alpha\beta} := \quad \{f : \mathcal{D}_\beta \to \mathcal{D}_\alpha \mid \forall p \in \mathcal{R}_\beta \ \boxed{\boxed{\langle f(p^0), f(p^1)\rangle}} \in \mathcal{R}_\alpha\}$

Define $\mathcal{R}_{\alpha\beta} := \{\boxed{\boxed{q \in (\mathcal{D}_{\alpha\beta})^2}} \mid \forall p \in \mathcal{R}_\beta \ \boxed{\boxed{\langle q^0(p^0), q^1(p^1)\rangle}} \in \mathcal{R}_\alpha\}$

Generalizing Logical Relation Frames

Start with $\mathcal{D}_o = \{\texttt{T}, \texttt{F}\}$, $\mathcal{D}_\iota$ nonempty, $\mathcal{R}_o \subseteq \mathcal{D}_o^{\boxed{2}}$ and $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^{\boxed{2}}$.

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Extend to function types:

Assume $\mathcal{D}_\alpha$, $\mathcal{R}_\alpha \subseteq (\mathcal{D}_\alpha)^{\boxed{2}}$, $\mathcal{D}_\beta$ and $\mathcal{R}_\beta \subseteq (\mathcal{D}_\beta)^{\boxed{2}}$ are defined.

Define $\mathcal{D}_{\alpha\beta} := \;\; \{f : \mathcal{D}_\beta \to \mathcal{D}_\alpha \;\big|\; \forall p \in \mathcal{R}_\beta \;\; \boxed{\langle f(p^i)\rangle_{i\in\{0,1\}}} \in \mathcal{R}_\alpha\}$

Define $\mathcal{R}_{\alpha\beta} := \{\boxed{q : \{0,1\} \to \mathcal{D}_{\alpha\beta}} \;\big|\; \forall p \in \mathcal{R}_\beta \;\; \boxed{\langle q^i(p^i)\rangle_{i\in\{0,1\}}} \in \mathcal{R}_\alpha\}$

Generalizing Logical Relation Frames

Start with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$, $\mathcal{D}_\iota$ nonempty, $\mathcal{R}_o \subseteq \mathcal{D}_o^{\boxed{A}}$ and $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^{\boxed{A}}$.

Assume $\mathcal{R}_o$ and $\mathcal{R}_\iota$ include constant functions.

Extend to function types:

Assume $\mathcal{D}_\alpha$, $\mathcal{R}_\alpha \subseteq (\mathcal{D}_\alpha)^{\boxed{A}}$, $\mathcal{D}_\beta$ and $\mathcal{R}_\beta \subseteq (\mathcal{D}_\beta)^{\boxed{A}}$ are defined.

Define $\mathcal{D}_{\alpha\beta} := \quad \{f : \mathcal{D}_\beta \to \mathcal{D}_\alpha \mid \forall p \in \mathcal{R}_\beta \;\; \boxed{\langle f(p^i)\rangle_{i\in A}} \in \mathcal{R}_\alpha\}$

Define $\mathcal{R}_{\alpha\beta} := \{\boxed{q : A \to \mathcal{D}_{\alpha\beta}} \mid \forall p \in \mathcal{R}_\beta \;\; \boxed{\langle q^i(p^i)\rangle_{i\in A}} \in \mathcal{R}_\alpha\}$

Generalizing Logical Relation Frames

Start with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$, $\mathcal{D}_\iota$ nonempty, $\mathcal{R}_o \subseteq \mathcal{D}_o^{\boxed{A}}$ and $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^{\boxed{A}}$.

Assume $\mathcal{R}_o$ and $\mathcal{R}_\iota$ include constant functions.

Extend to function types:

Assume $\mathcal{D}_\alpha$, $\mathcal{R}_\alpha \subseteq (\mathcal{D}_\alpha)^{\boxed{A}}$, $\mathcal{D}_\beta$ and $\mathcal{R}_\beta \subseteq (\mathcal{D}_\beta)^{\boxed{A}}$ are defined.

Define $\mathcal{D}_{\alpha\beta} := \quad \{f : \mathcal{D}_\beta \to \mathcal{D}_\alpha \mid \forall p \in \mathcal{R}_\beta \ \ \boxed{(f \circ p)} \in \mathcal{R}_\alpha\}$

Define $\mathcal{R}_{\alpha\beta} := \{\boxed{q : A \to \mathcal{D}_{\alpha\beta}} \mid \forall p \in \mathcal{R}_\beta \ \ \boxed{\langle q^i(p^i)\rangle_{i\in A}} \in \mathcal{R}_\alpha\}$

Generalizing Logical Relation Frames

Start with $\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$, $\mathcal{D}_\iota$ nonempty, $\mathcal{R}_o \subseteq \mathcal{D}_o^{\boxed{A}}$ and $\mathcal{R}_\iota \subseteq \mathcal{D}_\iota^{\boxed{A}}$.

Assume $\mathcal{R}_o$ and $\mathcal{R}_\iota$ include constant functions.

Extend to function types:

Assume $\mathcal{D}_\alpha$, $\mathcal{R}_\alpha \subseteq (\mathcal{D}_\alpha)^{\boxed{A}}$, $\mathcal{D}_\beta$ and $\mathcal{R}_\beta \subseteq (\mathcal{D}_\beta)^{\boxed{A}}$ are defined.

Define $\mathcal{D}_{\alpha\beta} := \quad \{f : \mathcal{D}_\beta \to \mathcal{D}_\alpha \,|\, \forall p \in \mathcal{R}_\beta \ \boxed{(f \circ p)} \in \mathcal{R}_\alpha\}$

Define $\mathcal{R}_{\alpha\beta} := \{\boxed{q : A \to \mathcal{D}_{\alpha\beta}} \,|\, \forall p \in \mathcal{R}_\beta \ \boxed{S(q, p)} \in \mathcal{R}_\alpha\}$

where $S(q, p)(i) := q(i)(p(i))$.

Logical Relation Frames

Given: A, $\mathcal{D}_\iota$, $\mathcal{R}_\iota \subseteq (\mathcal{D}_\iota)^A$, $\mathcal{R}_o \subseteq (\{\mathtt{T}, \mathtt{F}\})^A$

Assumptions:

- $\mathcal{D}_\iota$ is nonempty.
- $\mathcal{R}_\iota$ and $\mathcal{R}_o$ contain all constant functions.

Conclusion:

- The LR frame $\mathcal{D}$ is a combinatory frame.

A Special Case: Frames with Specified Sets

Let A be nonempty and $\mathcal{B} \subseteq \mathcal{P}A$. Assume $\emptyset \in \mathcal{B}$ and $A \in \mathcal{B}$.

Define $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$ and $\mathcal{D}_\iota := A$.

Define $\mathcal{R}_o \subseteq (\mathcal{D}_o)^A$: $\{p : A \to \mathcal{D}_o \mid p^{-1}(\mathtt{T}) \in \mathcal{B}\} = \{\chi_X \mid X \in \mathcal{B}\}$

Define $\mathcal{R}_\iota \subseteq (\mathcal{D}_\iota)^A$: $\{p : A \to \mathcal{D}_\iota \mid p^{-1}(X) \in \mathcal{B} \text{ whenever } X \in \mathcal{B}\}$

Extend $\mathcal{D}$ and $\mathcal{R}$ to all types giving combinatory frame $\mathcal{D}$.

Fact: $\mathcal{D}_{o\iota} = \{\chi_X | X \in \mathcal{B}\}$. ($\mathcal{D}_{o\iota} \cong \mathcal{B}$)

Idea: We *specified* $\mathcal{D}_\iota$ and $\mathcal{D}_{o\iota}$ by giving A and $\mathcal{B}$, then extended definition to all types using relations.

A Special Case: Frames with Specified Sets

Given: A, $\mathcal{B} \subseteq \mathcal{P}A$

Assumptions:

- A nonempty
- $\emptyset \in \mathcal{B}$ nonempty
- $A \in \mathcal{B}$ nonempty

Conclusion:

- Specified sets frame $\mathcal{D}$ is a combinatory frame where
- $\mathcal{D}_\iota = A$ and $\mathcal{D}_{o\iota} \cong \mathcal{B}$

Specified Sets and Logical Constants

- $\mathcal{D}$ realizes $\neg$ iff $\mathcal{B}$ is closed under complements.
- $\mathcal{D}$ realizes $\wedge$ iff $\mathcal{B}$ is closed under binary intersections.
- $\mathcal{D}$ realizes $\vee$ iff $\mathcal{B}$ is closed under binary unions.

Frame in which Surjective Cantor Fails

Let A be the real interval $(-1,1)$ and

$$\mathcal{B} := \{(a,1) \;\big|\; -1 \leq a \leq 1\} \subseteq \mathcal{P}(A).$$

Note: $\emptyset$ is $(1,1) \in \mathcal{B}$ and A is $(-1,1) \in \mathcal{B}$.

Define $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$ and $\mathcal{D}_\iota := A$.

Use Relations to define combinatory frame $\mathcal{D}$ with $\mathcal{D}_{o\iota} \cong \mathcal{B}$.

Facts:

- $\mathcal{D}_{o\iota} = \{\chi_{(a,1)} \;\big|\; -1 \leq a \leq 1\}$
- The negation function $\neg : \mathcal{D}_o \to \mathcal{D}_o$ is not in $\mathcal{D}_{oo}$.
- $\mathcal{D}$ realizes $\wedge$ and $\vee$.

Surjection in $\mathcal{D}_{o\iota}$

$\mathcal{D}_\iota = (-1, 1)$

$\mathcal{D}_{o\iota} = \{\chi_{(a,1)} \mid -1 \leq a \leq 1\}$

Define $G : \mathcal{D}_\iota \to \mathcal{D}_{o\iota}$ as follows:

$$G(x) := \begin{cases} \chi_{\emptyset} & \text{if } -1 < x \leq -\frac{1}{2} \\ \chi_{(-2x,1)} & \text{if } -\frac{1}{2} < x < \frac{1}{2} \\ \chi_{(-1,1)} & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

That is, $G(x)(y) = \mathtt{T}$ iff $-2x < y$ for each $x, y \in (-1, 1)$.

Fact 1: G is surjective.

Fact 2: $G \in \mathcal{D}_{o\iota\iota}$.

Injective Cantor Theorem

There is no injection from $\mathcal{D}_{o\iota}$ to $\mathcal{D}_{\iota}$.

$$\neg\exists\, h_{\iota(o\iota)} \forall\, p_{o\iota} \forall\, q_{o\iota} [h\, p =^{\iota} h\, q \supset p =^{o\iota} q]$$

Informal Proof: Assume $H_{\iota(o\iota)}$ is injective.

- Let $\mathbf{D}$ be $\{[H\, X_{o\iota}] | [H\, X] \notin X\}$ (diagonal set).
- Consider $[H\, \mathbf{D}]$. If $[H\, \mathbf{D}] \notin \mathbf{D}$, then $[H\, \mathbf{D}] \in \mathbf{D}$. Hence $[H\, \mathbf{D}] \in \mathbf{D}$.
- For some $X_{o\iota}$, $[[H\, \mathbf{D}] =^{\iota} [H\, X]]$ and $[H\, X] \notin X$.
- By injectivity of H, $[X =^{o\iota} \mathbf{D}]$ and so $[H\, \mathbf{D}] \notin \mathbf{D}$.

Contradiction.

Injective Cantor Theorem

The diagonal set $\{[H\, X_{o\iota}] | [H\, X] \notin X\}$ can be formally defined as

$$[\lambda y_\iota\, \exists X_{o\iota}\, [[y =^\iota H\, X] \wedge \neg[X\, [H\, X]]]]$$

if $\neg$, $\wedge$, $=^\iota$ and $\Sigma^{o\iota}$ are in the signature $\mathcal{S}$.

There is a combinatory frame $\mathcal{D}$ such that:

- The injective Cantor Theorem is false.
- The surjective Cantor Theorem is true.
- $\mathcal{D}$ realizes $\neg$ and $\wedge$.
- $\mathcal{D}$ realizes neither $=^\iota$ nor $\Sigma^{o\iota}$.

Frame in which Injective Cantor Fails

Let A be the natural numbers $\mathbb{N}$.

Let $\mathcal{B}$ be the set of finite $X \subseteq \mathbb{N}$ and cofinite $Y \subseteq \mathbb{N}$.

Note: $\emptyset \in \mathcal{B}$ (finite), $\mathbb{N} \in \mathcal{B}$ (cofinite), and $\mathcal{B}$ is closed under complements, binary unions and binary intersections.

Define $\mathcal{D}_o := \{\mathtt{T}, \mathtt{F}\}$ and $\mathcal{D}_\iota := \mathbb{N}$.

Use Relations to define combinatory frame $\mathcal{D}$ with $\mathcal{D}_{o\iota} \cong \mathcal{B}$.

$\mathcal{D}$ is a $\{\neg, \wedge, \vee\}$-model.

Frame in which Injective Cantor Fails

$\mathcal{D}_o = \{\mathtt{T}, \mathtt{F}\}$

$\mathcal{D}_\iota = \mathbb{N}$

$\mathcal{R}_o = \{\chi_X : \mathbb{N} \to \mathcal{D}_o \mid X \text{ finite or cofinite}\}$

Facts:

- $f \in \mathcal{D}_{\iota(o\iota)}$ for every $f : \mathcal{D}_{o\iota} \to \mathcal{D}_\iota$.
- Every injection from $\mathcal{D}_{o\iota}$ to $\mathcal{D}_\iota$ (both are countable) is in $\mathcal{D}_{\iota(o\iota)}$.
- The Injective Cantor Theorem fails in $\mathcal{D}$.

- $f \in \mathcal{D}_{o(o\iota)}$ for every $f : \mathcal{D}_{o\iota} \to \mathcal{D}_o$.
- Π^ι and Σ^ι have interpretations in $\mathcal{D}_{o(o\iota)}$.

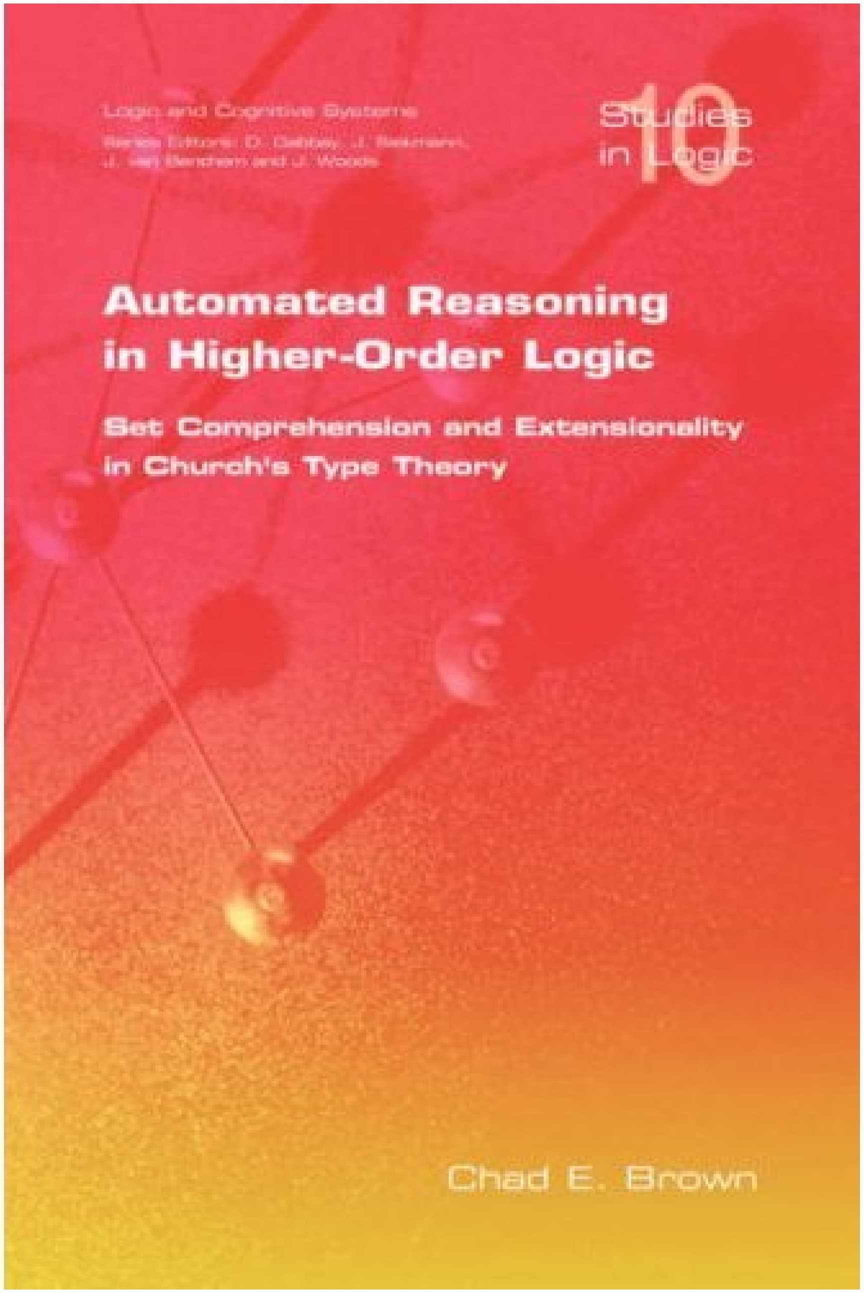
Power Types

Define types $\overline{n}$ as follows:

- $\overline{0} := \iota$ (individuals)
- $\overline{n+1} := (o\overline{n})$ (sets of $\overline{n}$)

Frame in which Injective Cantor Fails

- $\mathcal{D}_{\overline{0}}$ and $\mathcal{D}_{\overline{1}}$ are countable.
- $\mathcal{D}_{\overline{2}} = (\mathcal{D}_o)^{\mathcal{D}_{\overline{1}}}$ is uncountable (continuum).
- $\mathcal{D}_{\overline{3}}$ is again *countable*.
- $\mathcal{D}_{\overline{4}} = (\mathcal{D}_o)^{\mathcal{D}_{\overline{3}}}$ is uncountable (continuum).
- Alternates up the heirarchy.



These results are in my Ph.D. dissertation, now published as a book: Automated Reasoning in Higher-Order Logic

Speculation: LR and NF

LR frames easily give models of fragments of type theory where $\mathcal{D}_\iota$ and $\mathcal{D}_{o\iota}$ have the same cardinality.

Could we find a model of NF this way?

Specker's Result

- (1958) *Duality* Ernst Specker

(translated by Thomas Forster)

- **Specker's Result:** There is a model of NF iff there is a model of simply typed set theory with an ϵ-automorphism π mapping each type k one-to-one onto $k+1$.

Specker's result was relative to simply typed set theory:

Types: $\{0, 1, 2, \ldots\}$

Uses $\in$ instead of application.

Let's switch to this point of view.

Specker's Result

Idea: Given $(V, \in) \models NF$, choose each $\mathcal{D}_k := V$.

- Extensionality and (typed) comprehension principles hold.
- There is a bijection from each $\mathcal{D}_k$ onto $\mathcal{D}_{k+1}$ preserving $\in$.

Conversely: Given $\mathcal{D}_k$ for each satisfying extensionality and typed comprehension, and an $\in$-preserving bijection from each $\mathcal{D}_k$ onto $\mathcal{D}_{k+1}$, **then** $\mathcal{D}_0$ will model NF.

Specker's Result

Combining Specker's Result with $NF = NF_4$ it is enough to find a model of type theory with four types

$$\langle \mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \rangle$$

where

$$\mathcal{D}_{i+1} \subseteq \mathcal{P}(\mathcal{D}_i)$$

along with $\in$-preserving bijections

$$\pi_i : \mathcal{D}_i \longrightarrow \mathcal{D}_{i+1}$$

Using LR...

Given: A (nonempty), $\mathcal{B} \subseteq \mathcal{P}A$, $\pi : A \to \mathcal{B}$, $\mathcal{R}_0 \subseteq A^A$

Assume: $\emptyset \in \mathcal{B}$, $A \in \mathcal{B}$, $\mathcal{B}$ is a field of sets, π is bijective, $\mathcal{R}_0$ contains all constant functions.

Define:

$$\mathcal{D}_0 := A$$

$$\mathcal{D}_{i+1} := \{X \subseteq \mathcal{D}_i | \forall g \in \mathcal{R}_i g^{-1}(X) \in \mathcal{B}\}$$

$$\mathcal{R}_{i+1} := \{k : A \to \mathcal{D}_{i+1} | \forall g \in \mathcal{R}_i \{x \in A | g(x) \in k(x)\} \in \mathcal{B}\}$$

$$\pi_0 : \mathcal{D}_0 \to \mathcal{P}(\mathcal{D}_0) \text{ by } \pi_0 := \pi$$

$$\pi_{i+1} : \mathcal{D}_{i+1} \to \mathcal{P}(\mathcal{D}_{i+1}) \text{ by } \pi_{i+1}(X) := \{\pi_i(x) | x \in X\}$$

Using LR...

Want: $\mathcal{D}_1 = \mathcal{B}$.

The following conditions guarantee this:

(A): $\forall g \in \mathcal{R}_0 \forall X \in \mathcal{B}\, g^{-1}(X) \in \mathcal{B}$

(B): $id \in \mathcal{R}_0$

Now $\pi^0 : \mathcal{D}_0 \to \mathcal{D}_1$ is a bijection
(since $\pi : A \to \mathcal{B}$ was assumed a bijection).

Using LR...

Want $\pi^1 : \mathcal{D}_1 \to \mathcal{D}_2$ **bijective.**

The following conditions guarantee this:

(C^1) For $q \in \mathcal{R}_1$ and $X \in \mathcal{B}$,

$$\{x \in A | \exists y \in X(q(x) = \pi(y))\} \in \mathcal{B}$$

(D^1) For $W \in \mathcal{D}_2$, we have

$$\{x \in A | \pi(x) \in W\} \in \mathcal{B}$$

Using LR...

Want $\pi^2 : \mathcal{D}_2 \to \mathcal{D}_3$ **bijective.**

The following conditions guarantee this:

(C^2) For $Q \in \mathcal{R}_2$ and $W \in \mathcal{D}_2$,

$$\{x \in A | Q(x) \in \pi^2(W)\} \in \mathcal{B}$$

(D^2) For $P \in \mathcal{D}_3$, we have

$$\{X \in \mathcal{D}_1 | \pi^1(X) \in P\} \in \mathcal{D}_2,$$

Using LR...

Want comprehension
– equality must be realized for types $0, 1, 2, 3$**.**

The following conditions guarantee this:

(E^i) (where $i \in \{0, 1, 2, 3, 4\}$). For $g, h \in \mathcal{R}_i$,

$$\{x \in A | g(x) = h(x)\} \in \mathcal{B}$$

11 Conditions

Goal: Find A, $\mathcal{B}$, π, $\mathcal{R}_0$ satisfying:

(A): $\forall g \in \mathcal{R}_0 \forall X \in \mathcal{B}\, g^{-1}(X) \in \mathcal{B}$

(B): $id \in \mathcal{R}_0$

(C^1) For $q \in \mathcal{R}_1$ and $X \in \mathcal{B}$,

$$\{x \in A | \exists y \in X(q(x) = \pi(y))\} \in \mathcal{B}$$

(C^2) For $Q \in \mathcal{R}_2$ and $W \in \mathcal{D}_2$,

$$\{x \in A | Q(x) = \pi^2(W)\} \in \mathcal{B}$$

11 Conditions

(D^1) For $W \in \mathcal{D}_2$, we have

$$\{x \in A | \pi(x) \in W\} \in \mathcal{B}$$

(D^2) For $P \in \mathcal{D}_3$, we have

$$\{X \in \mathcal{D}_1 | \pi^1(X) \in P\} \in \mathcal{D}_2,$$

(E^i) (where $i \in \{0, 1, 2, 3, 4\}$). For $g, h \in \mathcal{R}_i$,

$$\{x \in A | g(x) = h(x)\} \in \mathcal{B}.$$

This would give a model of NF.

Formal Versions

A formal version of the problem is available on this web page:

`mathgate.info/nf/`

Potential solutions can be given as definitions for

$$A, \mathcal{B}, \pi, \mathcal{R}_0$$

and proof terms for the 11 main conditions (plus 10 smaller conditions – A nonempty, etc.) to be checked by a proof checker.

Converse Question

Given a model $(V, \in)$ of NF, can we choose $\mathcal{R}_0$ so that the 11 conditions hold taking

$$A = V$$

$$\mathcal{B} \cong V$$

and π to be the obvious bijection?

Conclusion

- Logical relations yield models of fragments of Church's type theory in which...
  - ...Surjective Cantor Fails
  - ...Injective Cantor Fails
- I have given sufficient conditions to obtain a model of NF using the techniques.
  - Are they consistent?
  - Are they necessary?